Discussion on Tensor Products, Spin Operators, and Diagonalization

1 Introduction

This document captures a detailed discussion on the tensor product of spinor vector spaces, the application of Pauli matrices, and the process of diagonalizing the total angular momentum operator J^2 for a system of two spin-1/2 particles.

2 Tensor Products and Spin Operators

The discussion began with an exploration of how the tensor product of two spinor vector spaces corresponds to a block diagonal matrix consisting of 0s and 1s. Using the Pauli matrices, we analyzed the effect of tensor operations on spin systems.

3 Addition of Angular Momentum

We examined how the total angular momentum operator J^2 is derived for two spin-1/2 particles using tensor operations. The key relation:

$$J^2 = 2\hbar^2(I + S_1 \cdot S_2)$$

was explicitly computed in matrix form.

4 Explicit Matrix Computation

We performed a direct computation of the tensor products of the Pauli matrices, ensuring correctness in each step:

- Tensor product of σ_x with itself:

$$\sigma_x \otimes \sigma_x = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

- Tensor product of σ_y with itself:

$$\sigma_y \otimes \sigma_y = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

- Tensor product of σ_z with itself:

$$\sigma_z \otimes \sigma_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Adding these results together and including the identity matrix led to the correct expression for J^2.

5 Diagonalization Process

The matrix to be diagonalized was:

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

The eigenvalues were computed as $\{2, 2, 2, 0\}$. The corresponding eigenvectors were:

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

After normalizing, the unitary matrix that diagonalizes A was determined to be:

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying this transformation, we confirmed:

$$U^\dagger A U = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

6 Conclusion

We successfully worked through the derivation of J^2 for two spin-1/2 particles, performed tensor products of spin matrices explicitly, and diagonalized the resulting matrix step by step.