Tutorial to Develop the theory of Addition of Angular Momentum Through the Investigation of the D_s^+ and D_s^{*+} Mesons

B. Toggerson
University of Massachusetts Amherst
Physics 564

Spring 2025

1 Angular Momentum Toolkit

1.1 Notation

Let $\vec{J}$ be the sum of two other angular momenta which, for the purposes of notational clarity, we will call $\vec{L}$ and $\vec{S}$; i.e.,

$$\vec{J} = \vec{L} + \vec{S}.$$

Both $\vec{L}$ and $\vec{S}$ may refer to either spin or orbital angular momentum depending on the context. Similarly, we will use $\mathcal{J}$ for an arbitrary angular momentum.

Kets will be written as:

- $|j, m_j\rangle$ for a single angular momentum.
- $|\ell, s, m_\ell, m_s\rangle$ for combined angular momentum in the uncoupled basis, which, since ℓ and s are generally fixed values, we will shorten to $|m_\ell, m_s\rangle$.
- $|j, \ell, s, m_j\rangle$ for combined angular momenta in the coupled basis. Again, since ℓ and s are generally fixed, we will shorten this to $|j, m_j\rangle$.

Finally, we will use $|\uparrow\rangle$ to refer to a spin-1/2 particle in the $s_z = +\hbar/2$ eigenstate and $|\downarrow\rangle$ to refer to a spin-1/2 particle in the $s_z = -\hbar/2$ eigenstate.

1.2 Fundamental Operators

$$\mathcal{J}^2|j,m_j\rangle = j(j+1)\hbar^2|j,m_j\rangle \quad \mathcal{J}_z|j,m_j\rangle = m_j\hbar|j,m_j\rangle$$

$$\mathcal{J}_\pm = \mathcal{J}_x \pm i\mathcal{J}_y$$

$$\mathcal{J}_\pm|j,m_j\rangle = \sqrt{j(j+1) - m_j(m_j \pm 1)}\hbar|j,(m_j \pm 1)\rangle$$

1.3 Commutators and Other Properties

In General

$$[\mathcal{J}^2, \mathcal{J}_i] = 0 \quad [\mathcal{J}_i, \mathcal{J}_j] = i\epsilon_{ijk}\hbar\mathcal{J}_k$$

For $\vec{J} = \vec{L} + \vec{S}$

$$J_\pm = L_\pm + S_\pm \quad J^2 = L^2 + S^2 + 2\vec{L}\cdot\vec{S} = L^2 + S^2 + 2L_zS_z + L_+S_- + L_-S_+$$

$$[J^2, L^2] = [J^2, S^2] = [L_i, S_i] = [L^2, S_i] = [L_i, S^2] = [L\pm, S\pm] = [J_\pm, L_\pm] = [J_\pm, S_\pm] = 0$$

$$[J^2, \vec{L}] = 2i\hbar\left(\vec{L} \times \vec{S}\right) \neq 0$$

2 Tutorial

For this tutorial we will be considering the mesons with which have the quark content $c\bar{s}$ (aka a charm and an anti-strange quark) and in which the quarks have zero angular momentum about each other. We are using the c and $\bar{s}$ quarks so we don't have the words "up" and "down" referring to both spin and quark content! Both quarks are spin-1/2 particles. Call the spin of the charm $\vec{L}$ and the spin of the anti-strange $\vec{S}$.

As we saw last time, in the uncopled basis $|\ell_z s_z\rangle$ we have four states:

1. $|\uparrow\uparrow\rangle$
2. $|\uparrow\downarrow\rangle$
3. $|\downarrow\uparrow\rangle$
4. $|\downarrow\downarrow\rangle$

However, these states are not eigenstates of J^2 as $[J^2, \vec{L}] \neq 0$. We want to find linear combinations of these states which *are* eigenstates of J^2.

2.1 Understanding the *Uncoupled* Representation First!

For each of the uncoupled states, determine L_z, S_z, J_z, L^2 and S^2:

State	L_z	S_z	J_z	L^2	S^2
$\vert\uparrow\uparrow\rangle$					
$\vert\uparrow\downarrow\rangle$					
$\vert\downarrow\uparrow\rangle$					
$\vert\downarrow\downarrow\rangle$					

2.2 Now to Generate Eigenstates of J^2

1. We see that there is only one state with $J_z = \hbar$: $|\uparrow\uparrow\rangle$. Figure out $J^2|\uparrow\uparrow\rangle$. Is this is an eigenstate of J^2? If so, to what value of j does this state correspond?

2. Now, we can get the next state by applying J_-:

   (a) Determine the result of $J_-|j, m_j\rangle = J_-|\uparrow\uparrow\rangle$

   (b) What is J_z for this state?

   (c) Apply J^2 to the state you obtained in part (a). What is the result and value of j? Write the corresponding coupled ket $|j, m_j\rangle$

3. Now do J_- *again*:

   (a) Determine the result of applying J_- on the result from Problem 2.

   (b) What is J_z for this state?

   (c) Apply J^2 to the state you obtained in part 4.(a). What is the result and value of j? Write the corresponding coupled ket $|j, m_j\rangle$

4. Now do J_- *again*:

   (a) Determine the result of applying J_- on the result from Problem 3.

5. **Degree of freedom check!** How many independent states have we found? How many were in our uncoupled representation? Is there a problem?

6. One way to find another degree of freedom is to find a coupled state $|j, m_j\rangle$ which, when represented in the uncoupled basis $|m_\ell, m_s\rangle$ is orthogonal to one of the ones we have already found.

   (a) For which of the state(s) which are eigenstates of J^2 (i.e. those you found in parts 2, 3, and 4), is it *possible* to construct an orthogonal state in the *uncoupled* basis?

   (b) Build that orthogonal state.

   (c) Apply J_z to determine m_j of this state.

(d) Apply J^2 to determine j of this state.

(e) What happens if you apply J_- to this state?

7. **Degree of freedom check!** How many states with do you have for each value of m_j do you have in the coupled basis? In the uncoupled basis? Is everything okay now?

8. **Summary:** If you add $\vec{L} + \vec{S} = \vec{J}$ with L and S both being spin-1/2 particles:

   (a) What possible values of j are allowed?

   (b) Write down the J_z matrix/matricies.

9. **Application:** There are two mesons with the quark content $c\bar{s}$ in which the two quarks have zero orbital angular momentum about each other: D_s^+ which has a spin =0, and D_s^{*+} which has a spin = 1.

   (a) Write down the possible state(s) for each meson in terms of the uncoupled representation of the charm and anti-strange spins.

   (b) Which do you expect would be more massive?

10. **Generalize:** If you added a spin-1 and a spin-1/2 particle together, what value(s) of j would be possible?