Making S_y by Rotating S_z

Last time we saw

$$|\uparrow\rangle_x = \frac{1}{\sqrt{2}}|\uparrow\rangle_z + \frac{1}{\sqrt{2}}|\downarrow\rangle_z$$

and

$$|\downarrow\rangle_x = \frac{1}{\sqrt{2}}|\uparrow\rangle_z - \frac{1}{\sqrt{2}}|\downarrow\rangle_z$$

which means

$$\sqrt{2}\,|\uparrow\rangle_z = |\uparrow\rangle_x + |\downarrow\rangle_x$$

$$|\uparrow\rangle_z = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_x + |\downarrow\rangle_x\right)$$

and

$$|\downarrow_z\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_x - |\downarrow\rangle_x\right)$$

$$\langle\uparrow|\downarrow\rangle_z = \left[\frac{1}{\sqrt{2}}\left(\langle\uparrow|_x + \langle\downarrow|_x\right)\right]\left[\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_x - |\downarrow\rangle_x\right)\right]$$

$$= \frac{1}{2}\left[\langle\uparrow|\uparrow\rangle_x - \cancel{\langle\uparrow|\downarrow\rangle_x} + \cancel{\langle\downarrow|\uparrow\rangle_x} - \langle\downarrow|\downarrow\rangle_x\right]$$

by $\perp$ & add to zero!

$$= \frac{1}{2}[1-1] = 0 \quad \text{as it should}$$

We similarly saw that

$$|\uparrow\rangle_y = \frac{1}{\sqrt{2}}\left[\,|\uparrow\rangle_z + i\,|\downarrow\rangle_z\right]$$

and

$$|\downarrow\rangle_y = \frac{1}{\sqrt{2}}\left[\,|\uparrow\rangle_z - i\,|\downarrow\rangle_z\right]$$

Now, let's return to our ideas of angular momentum as the generator of rotations to do this another way:

Say $|\psi\rangle = |\uparrow\rangle_z$

We'll take $\varphi \to \pi/2$ later

Now, let's rotate by φ about $\tilde{x}$ to $|\psi'\rangle$

Ex: Write the expression

$$|\psi'\rangle = \exp\left[-\frac{i\varphi}{\hbar}\,\vec{S}\cdot\hat{n}\right]|\psi\rangle$$

in our case $\hat{n} = \hat{y}$

$$\therefore\ |\psi'\rangle = \exp\left[-\frac{i\varphi}{\hbar}\,S_x\right]|\psi\rangle$$

$$= \exp\left[-\frac{i\varphi}{\cancel{\hbar}}\,\frac{\cancel{\hbar}}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right]|\uparrow\rangle$$

Expand the exponential as that is the definition of e^{M}:

$$|\psi'\rangle = \left[1 + \left(\frac{-i\varphi}{2}\right)\begin{pmatrix}0&1\\1&0\end{pmatrix} + \frac{1}{2!}\left(\frac{-i\varphi}{2}\right)^2\begin{pmatrix}0&1\\1&0\end{pmatrix}^2 + \frac{1}{3!}\left(\frac{-i\varphi}{2}\right)^3\begin{pmatrix}0&1\\1&0\end{pmatrix}^3 + \frac{1}{4!}\left(\frac{-i\varphi}{2}\right)^4\begin{pmatrix}0&1\\1&0\end{pmatrix}^4 + \cdots\right]|\uparrow\rangle_z$$

we need the z-basis representation of S_y as that is the basis of our ket (Our ket is in the S_z vector-space).

Note $\begin{pmatrix}0&1\\1&0\end{pmatrix}^2 = \mathbb{1}_{2\times2}$ (true for all σ_i's!)

$\therefore \sigma_x^{2n+1} = \sigma_x$

Writing $|\uparrow\rangle_z$ as $\begin{pmatrix}1\\0\end{pmatrix}$ to be more explicit:

$$|\psi'\rangle = \begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{1!}\left(\frac{-i\varphi}{2}\right)\begin{pmatrix}0\\1\end{pmatrix} + \frac{1}{2!}\left(\frac{-i\varphi}{2}\right)^2\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{3!}\left(\frac{-i\varphi}{2}\right)^3\begin{pmatrix}0\\1\end{pmatrix} + \frac{1}{4!}\left(\frac{-i\varphi}{2}\right)^4\begin{pmatrix}1\\0\end{pmatrix} + \cdots$$

Or

$$|\psi'\rangle = |\uparrow\rangle + \left(\frac{-i\varphi}{2}\right)|\downarrow\rangle + \frac{1}{2!}\left(\frac{-i\varphi}{2}\right)^2|\uparrow\rangle + \frac{1}{3!}\left(\frac{-i\varphi}{2}\right)^3|\downarrow\rangle + \frac{1}{4!}\left(\frac{-i\varphi}{2}\right)^4|\uparrow\rangle + \cdots$$

$$|\psi'\rangle = \left[1 - \frac{1}{2!}\left(\frac{-i\varphi}{2}\right)^2 + \frac{1}{4!}\left(\frac{-i\varphi}{2}\right)^4 - \dots\right]|\uparrow\rangle$$

$$+\left[-i\left(\frac{\varphi}{2}\right) + i\frac{1}{3!}\left(\frac{\varphi}{2}\right)^3 - \dots\right]|\downarrow\rangle$$

$$|\psi'\rangle = \cos\left(\frac{\varphi}{2}\right)|\uparrow\rangle - i\sin\left(\frac{\varphi}{2}\right)|\downarrow\rangle$$

$$\varphi \to \pi/2$$

$$|\psi'\rangle = \cos\left(\frac{\pi}{4}\right)|\uparrow\rangle - i\sin\left(\frac{\pi}{4}\right)|\downarrow\rangle$$

$$= \frac{1}{\sqrt{2}}\left[|\uparrow\rangle - i|\downarrow\rangle\right] = |\downarrow\rangle_y \quad \text{o.ε.δ.}$$

Addition of Angular Momentum

- Talk about orbital and spin so have somthing to speak in
- $\vec{J} = \vec{L} + \vec{S}$
  - $\rightarrow J_y = L_y + S_y$
  - $\rightarrow J_x = L_x + S_x$
  - $\rightarrow J_z = L_z + S_z$

- We can tell if $\vec{J}$ is an angular momentum ~~because~~ if it follows commutations
  - $\rightarrow [J_x, J_y] = [L_x + S_x, L_y + S_y]$
  $= [L_x, L_y] + \cancel{[L_x, S_y]} + \cancel{[L_y, S_x]} + [S_x, S_y]$
  * because we can know S_y and L_x
  - $\rightarrow [J_x, J_y] = i\hbar L_z + i\hbar S_z = i\hbar J_z$
- All raising / lowering etc. still work
- $\vec{J}^2 \equiv J_x^2 + J_y^2 + J_z^2 = L^2 + 2\vec{L}\cdot\vec{S} + S^2$
- Lets play: Hydrogenic e^- in 2p state
  $\rightarrow \ell = 1$
  $\rightarrow$ 2 not really needed
- $[J^2, S_z] \overset{?}{=}$
  $\rightarrow [L^2 + (2L_xS_x + 2L_yS_y + 2L_zS_z) + S^2, S_z]$
  commute
  do not commute
  - $\rightarrow \therefore [J^2, S_z] \neq 0$
  - $\rightarrow$ Similarly $[J^2, L_z] \neq 0$
- $[J^2, L^2] = 0$
- $[J^2, S^2] = 0$
- Could have basis states $|\ell\ s\ m_\ell\ m_s\rangle$
  - $\rightarrow$ ℓ and s often omitted because they are obvious
- Or could write $|\ell\ s\ j\ m_j$

Adding 2 Angular Momenta

- All angular momenta work the same
- Call one L and one S for argument
- Recall that:
  - $\to J^2, J_z, L^2, S^2$ commute
  - $\to L_z$ and S_z do not commute with J^2
  - $\to L^2, S^2, L_z, S_z$ all commute

2 Distinguishable Spin 1/2 particles

- $S_1^2 |\psi\rangle = \hbar^2(\tfrac{1}{2})(\tfrac{1}{2}+1)|\psi\rangle$
- $m_1 = \tfrac{1}{2}$ or $-\tfrac{1}{2}$
- $m_2 = \tfrac{1}{2}$ or $-\tfrac{1}{2}$
- Can write a wave function as $|S_1^2\ S_2^2\ S_{1z}\ S_{2z}\rangle$
  - $\to = |\tfrac{1}{2}\ \tfrac{1}{2}\ \pm\tfrac{1}{2}\ \pm\tfrac{1}{2}\rangle$ will ignore
  - $\to$ since S_1^2 and S_2^2 always $= \tfrac{1}{2}$
  - $\to$ use $\uparrow\downarrow$'s for $+\tfrac{1}{2}$ and $-\tfrac{1}{2}$ respectively
- Basis set (uncoupled) $\left(|\psi\rangle = \alpha|\uparrow\uparrow\rangle + \beta|\uparrow\downarrow\rangle + \gamma|\downarrow\uparrow\rangle + \delta|\downarrow\downarrow\rangle\right)$
  - $\to |\uparrow\uparrow\rangle \quad S_z = +1$
  - $\to |\uparrow\downarrow\rangle \quad S_z = 0$
  - $\to |\downarrow\uparrow\rangle \quad S_z = 0$
  - $\to |\downarrow\downarrow\rangle \quad S_z = -1$
- S_z is a diagonal matrix since this basis

$$\hat{S}_z = \hbar\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$+ \Rightarrow \hbar\sqrt{l(l+1) - m(m+1)}$

$- \Rightarrow \hbar\sqrt{l(l+1) - m(m-1)}$

$\frac{3}{4} = \frac{1}{2}(\frac{1}{2}+1)$

$\frac{3}{4} = \frac{3}{4}$

- What about $\vec{S}^2$?
  - $\to$ $\vec{S}^2 = (\vec{S}_1 + \vec{S}_2)^2$ ← commuting not a problem
  - $\to$ $\vec{S}^2 = (\vec{S}_1 \cdot \vec{S}_1 + 2\vec{S}_1 \cdot \vec{S}_2 + \vec{S}_2 \cdot \vec{S}_2)$
  - $\to$ $\vec{S}^2 = S_1^2 + S_2^2 + 2(S_{1z}S_{2z} + \underbrace{S_{1x}S_{2x} + S_{1y}S_{2y}}_{\text{Identity } \frac{1}{2}(S_{1+}S_{2-} + S_{1-}S_{2+})})$
  - $\to$
  - $\to$ $S^2 = S_1^2 + S_2^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+}$
  - $\to$ good form because we know how all of these operators act on ψ's

Ex
- $S^2|\uparrow\uparrow\rangle$
  - $\to$ $(S_1^2 + S_2^2 + 2S_{1z}S_{2z} + S_{1+}S_{2-} + S_{1-}S_{2+})|\uparrow\uparrow\rangle$
  - $\to$ $\hbar^2[\frac{3}{4}|\uparrow\uparrow\rangle + \frac{3}{4}|\uparrow\uparrow\rangle + \frac{1}{2}|\uparrow\uparrow\rangle + 0 + 0]$
  - $\to$ $\hbar^2 2|\uparrow\uparrow\rangle$
  - $\to$ $s(s+1) = 2$
  - $\to$ $s = 1$ ← wow $\frac{1}{2}+\frac{1}{2}$!
  - $\to$ $\therefore$ $|\uparrow\uparrow\rangle$ has:
    - * $S_1 = 1/2$
    - * $S_2 = 1/2$
    - * $m_1 = 1/2$
    - * $m_2 = 1/2$
    - * AND $s = 1$
- Similarly for $|\uparrow\downarrow\rangle$:
  - $\to$ $S^2|\uparrow\downarrow\rangle =$
  - $\to$ $= \hbar^2[\frac{3}{4}|\uparrow\downarrow\rangle + \overbrace{\frac{3}{4}}^{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}|\uparrow\downarrow\rangle - \frac{1}{2}|\uparrow\downarrow\rangle + 0 + |\downarrow\uparrow\rangle]$
  - $\to$ This is not an eigenstate of S^2!! b/c $\to$
- For $|\downarrow\uparrow\rangle$ S^2 comes out to be $|\downarrow\uparrow\rangle + |\uparrow\downarrow\rangle$

• In a matrix

$$S^2 = \hbar^2 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

→ We need eigenvectors and eigenvalues for

* done before

* $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ $\lambda = 2$ $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$ $\lambda = 0$

* These λ's can be used via $\lambda = s(s+1)$ to get the s values for these states

→ S^2 eigenstates

$\lvert\uparrow\uparrow\rangle$	$s=1$	$s_z = 1$
$\frac{1}{\sqrt{2}}\lvert\uparrow\downarrow\rangle + \frac{1}{\sqrt{2}}\lvert\uparrow\downarrow\rangle$	$s=1$	$s_z = 0$
$\frac{1}{\sqrt{2}}\lvert\uparrow\downarrow\rangle - \frac{1}{\sqrt{2}}\lvert\downarrow\uparrow\rangle$	$s=0$	$s_z = 0$
$\lvert\downarrow\downarrow\rangle$	$s=1$	$s_z = -1$

from applying S_z matrix to state values are in order on other sheet

General Angular Momentum Addition

• $j_1 \otimes j_2$

$j_1 \rightarrow (2j_1+1)$ states

$j_2 \rightarrow (2j_2+1)$ states

→ a total $(2j_1+1)(2j_2+1)$ states

j_1 : $m_1 = -j_1, -j_1+1, \ldots\ldots\ldots +j_1$
j_2 $m_2 = -j_2, -j_2+1, \ldots\ldots +j_2$
maximum ~~$m_j = j_1+j_2$~~ $m_j = j_1+j_2$
$\therefore m_j \leq j_1+j_2$

$\left| \begin{matrix} m_1 & m_2 \\ j_1 & j_2 \end{matrix} \right\rangle$ = only way to get $m = j_1+j_2$

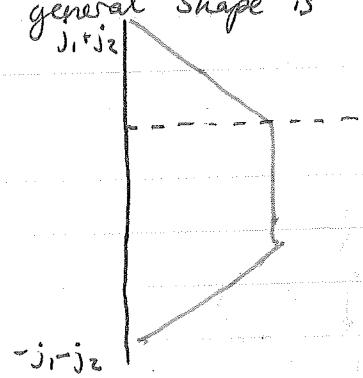
- Commuting operators:
  - Uncoupled J_1^2 J_{1z} J_2^2 J_{2z}
  - Coupled J_1^2 J_2^2 J^2 J_z
  - <u>BUT</u> $[J^2, J_{1z}] \neq 0$
  - I know I can find a basis of eigenstates of coupled $\mathcal{V}$
  - eigenstates of coupled $\mathcal{V}$ <u>must</u> be linear combinations of uncoupled states
  - uncoupled states are just lists of possibilities
- Eigenstates of J^2 come in multiplets of $m = -\hbar j, \ldots \hbar j$ which has $(2j+1)$ elements
- Cannot have $j > j_1+j_2$ as that would imply an m greater than j_1+j_2 which we have shown you cannot do
  - Similarly j cannot be $< j_1+j_2$ because then $\nexists$ an $m = j_1+j_2$
  - In fact can only have <u>exactly</u> <u>one</u> multiplet with $j_1+j_2 = j$.

$m \equiv m_1 + m_2$	$\vert \; m_1 \quad m_2 \; \rangle$	
$j_1 + j_2$	$\vert j_1, j_2 \rangle$	(1 state)
$j_1 + j_2 - 1$	$\vert (j_1 - 1), j_2 \rangle$ or $\vert j_1, (j_2 - 1) \rangle$	(2 states)
$j_1 + j_2 - 2$	$\vert (j_1 - 2), j_2 \rangle$; $\vert j_1, (j_2 - 2) \rangle$; $\vert (j_1 - 1), (j_2 - 2) \rangle$	(3 states)
$j_1 + j_2 - 3$	(4 states)	
$j_1 + j_2 - 4$	(5 states)	

→ until $j_1 - X = -j_1$ or $j_2 - X = -j_2$

* i.e. cannot lower one ~~anymore~~ anymore

→ general shape is



where one bottoms out @ $|j_1 - j_2|$

• $j_1 \otimes j_2 = j_1 + j_2 \oplus j_1 + j_2 - 1 \oplus j_1 + j_2 - 2 \oplus \dots \oplus |j_1 - j_2|$

↑ just list elements

In Cartesian Coordinates

• $l_1 = 1 \quad l_2 = 1$

• $l = 2, 1, 0$ with 9 states

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \otimes \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix} = \begin{pmatrix} p_x q_x \\ p_x q_y \\ p_x q_z \\ p_y q_x \\ \vdots \\ p_z q_z \end{pmatrix}$$

Conjecture: · When $l=1$ need 3 possibilityes so $\vec{p} \times \vec{q}$

Conjecture: · when $l=0$ need 1 possibility so $\vec{p} \cdot \vec{q}$

When $l=2$

$$= \frac{1}{3}\, p \cdot q\, \mathbb{1}_{3\times 3} + \begin{pmatrix} 0 & p_x q_y - p_y q_x & p_x q_z - p_z q_x \\ & 0 & \\ & 0 & 0 \end{pmatrix}$$