# Angular Momentum Review[1]

Toggerson Physics 564
University of Massachusetts Amherst

## Notation, Conventions, Summary and Toolkit

- The canonical angular momentum commutation relation is

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$$

- For a spin-one-half (spin-1/2) system, the spin quantum number $s = 1/2$ and the quantum number for the $z$-component of the spin is $m_z = \pm 1/2$. Therefore, if we choose the orthonormal basis vectors to be the eigenstates of the operators $S^2$ and $S_z$, we can denote them as $|s, m_z\rangle$:

$$\left|\frac{1}{2}, +\frac{1}{2}\right\rangle \text{ and } \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$

- Since the spin quantum number $s = 1/2$ is fixed, we can simplify the notation for the basis vectors as

$$|s, m_z\rangle = |m_z\rangle = \left|\pm\frac{1}{2}\right\rangle$$

- Another common notation for the basis vectors $|s, m_z\rangle$ for a spin-1/2 system is
    - $\left|+\frac{1}{2}\right\rangle = |\uparrow\rangle$
    - $\left|-\frac{1}{2}\right\rangle = |\downarrow\rangle$
- The eigenvalue equations for the operators $S_z$ and $S^2$ are given by:
    - $S_z|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle$ and $S_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle$
    - $S^2|\uparrow\rangle = \hbar^2 s(s+1)|\uparrow\rangle \xrightarrow{s=1/2} S^2|\uparrow\rangle = \frac{3\hbar^2}{4}|\uparrow\rangle$
    - $S^2|\downarrow\rangle = \hbar^2 s(s+1)|\downarrow\rangle \xrightarrow{s=\frac{1}{2}} S^2|\downarrow\rangle = \frac{3\hbar^2}{4}|\downarrow\rangle$
- In the algebra of angular momentum, the raising and lowering operators have the following effect on the simultaneous eigenstates of $S^2$ and $S_z$:

$$S_\pm|s, m_z\rangle = \hbar\sqrt{s(s+1) - m_z(m_z \pm 1)}\,|s, (m_z \pm 1)\rangle$$

    - The raising operator acting on the highest $m_z$ state annihilates it.
    - The raising operator acting on the lowest $m_z$ state annihilates it.
    - $S_\pm = S_x \pm iS_y$

[1] This tutorial is based on the Product Space Warmup Pretest and Spin Concept Tests 1-3 from the QUILTs: https://www.physport.org/curricula/quilts/

## Questions

For a spin-1/2 system, the spin quantum number $s = 1/2$ and the quantum number for the $z$ component of the spin is $m_z = \pm 1/2$. The matrix representations for $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$, which are the eigenstates of $\hat{S}_z$, are $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, respectively. Choose all of the following statements that are true.

(I) $\left|\frac{1}{2},\frac{1}{2}\right\rangle = |1/2\rangle = |\uparrow\rangle_z$

(II) $\left|-\frac{1}{2},-\frac{1}{2}\right\rangle = |-1/2\rangle = |\downarrow\rangle_z$

(III) $\hat{S}_z|\downarrow\rangle_z = \hbar/2|\downarrow\rangle_z$

The matrix representation for $|\uparrow\rangle_z$, which is an eigenstate of $\hat{S}_z$, is $\begin{pmatrix}1\\0\end{pmatrix}$. Which of the following statements are correct about $|\uparrow\rangle_z$?

(I) $|\uparrow\rangle_z = \frac{1}{\sqrt{2}}(|\uparrow\rangle_x + |\downarrow\rangle_x)$

(II) $|\uparrow\rangle_z = \frac{1}{\sqrt{2}}(|\uparrow\rangle_x - |\downarrow\rangle_x)$

(III) $|\uparrow\rangle_z = \frac{1}{\sqrt{2}}(|\uparrow\rangle_y - |\downarrow\rangle_y)$

At time $t=0$, the initial state of a spin-1/2 particle is $|\uparrow\rangle_z$. Choose all of the following statements that are correct.

(1) If we perform a measurement of the $x$ or $y$ component of the spin at $t=0$, we will obtain zero for both.

(2) If we measure $\hat{S}_y$, we can get either $\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$ with equal probability.

(3) If we measure $\hat{S}^2$, we can only get $\frac{3\hbar^2}{4}$.

Suppose the initial spin state of an electron is $a|\uparrow\rangle_z + b|\downarrow\rangle_z$ where $a$ and $b$ are non-zero constants. Choose all of the following statements that are correct.

(1) If we measure the $z$-component of the spin of the electron, we can obtain $+\hbar/2$ with a probability $|a|^2$.

(2) If we measure the $x$-component of the spin of the electron, we can obtain $+\hbar/2$ with a probability $|a|^2$.

(3) If we measure the $x$-component of the spin of the electron, we can obtain $+\hbar/2$ with a probability $\frac{1}{2}|a+b|^2$.

Suppose the particle is initially in an eigenstate of the $x$-component of spin angular momentum operator $\hat{S}_x$. Choose all of the following statements that are correct:

(1) The expectation value $\langle S_x \rangle$ depends on time.

(2) The expectation value $\langle S_y \rangle$ depends on time.

(3) The expectation value $\langle S_z \rangle$ depends on time.

Suppose the particle is initially in an eigenstate of the $z$-component of spin angular momentum $\hat{S}_z$. Choose all of the following statements that are correct:

(1) The expectation value $\langle S_x \rangle$ depends on time.

(2) The expectation value $\langle S_y \rangle$ depends on time.

(3) The expectation value $\langle S_z \rangle$ depends on time.

Choose all of the following statements that are true about the Hamiltonian $H_0 = \text{constant } S_z$ for a spin-1/2 system:

1. If we choose two different bases (coordinates), $H_0$ may be a diagonal matrix in one basis but not in the other.
2. If the basis vectors are eigenstates of $H_0$, $H_0$ will be a diagonal matrix.
3. No matter what basis we choose, $H_0$ must always be a diagonal matrix by definition.

What is the matrix representation of $S^2$ if we choose the basis vectors in the order $\{|\uparrow\rangle, |\downarrow\rangle\}$ to construct the matrix.

What is the correct expression for $\langle\downarrow|S_-|\uparrow\rangle$ for a spin-1/2 system?

Consider a single spin-1/2 system (*e.g.*, an electron spin). What is the dimensionality of the vector space associated with the spin degrees of freedom?

(a) 2-dimensional
(b) 3-dimensional
(c) 4-dimensional
(d) finite dimensional but can be anything
(e) infinite dimensional

Consider a single spin-1 system (*e.g.*, spin of the $W$ boson). What is the dimensionality of the vector space associated with the spin degrees of freedom?

(a) 2-dimensional
(b) 3-dimensional
(c) 4-dimensional
(d) finite dimensional but can be anything
(e) infinite dimensional

Consider a single spin-3/2 system. What is the dimensionality of the vector space associated with the spin degrees of freedom?

(a) 2-dimensional
(b) 3-dimensional
(c) 4-dimensional
(d) finite dimensional but can be anything
(e) infinite dimensional

Write $S_z$ for a spin-3/2 system.