Thus, any rotation becomes

$$R(\alpha,\beta,\gamma) = R_{z'}(\gamma)\, R_{y'}(\beta)\, R_z(\alpha)$$

- $R_{y'}(\beta) = R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha)$
  - → Undo the z rotation bringing $y' = y$
  - → Rotate about $y = y'$ by β
  - → Re-rotate z
- $R_{z'}(\gamma) = R_{y'}(\beta) R_z(\gamma) R_{y'}^{-1}(\beta)$
- Put all back together → $R(\alpha,\beta,\gamma) = R_z(\alpha) R_y(\beta) R_z(\gamma)$

$\boxed{\text{Key Takeaway:}}$ You can describe any rotation with a 3x3 real, orthogonal matrix!

Rotations of vectors are SO(3)

1) Any rotation can be represented by a 3x3 matrix and the combination of any two rotations is itself a rotation (see the Euler angles above) → **closed ✓**
2) I can do nothing → **Identity represented by $I_{3\times3}$ ✓**
3) There is an inverse (again see discussion of Euler angles above) → **Inverse ✓**
4) We have already been in a representation of 3x3 matricies and matrix multiplication is associative
   ↳ **Associativity ✓**

What is the dimensionality?

For discrete groups, dim(G) was simply the number of elements.

But there are an infinity of SO(3) matricies

Or are there? How many independent components?

$$R = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$ Looks like $n^2 = 9$, but

$$R^T R = 1_{3\times3} \Rightarrow \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$\hookrightarrow$
$$a^2 + d^2 + g^2 = 1 \qquad ab + de + gh = 0$$
$$b^2 + e^2 + h^2 = 1 \qquad bc + ef + hi = 0$$
$$c^2 + f^2 + i^2 = 1 \qquad ac + df + gi = 0$$

9 parameters − 6 eqn = 3 d.o.f. which we already knew

- α, β, γ Euler angles
- Pitch, roll, yaw

In general $\dim(SO(N)) = \dfrac{N(N-1)}{2}$

This dimensionality as number of free parameters is <u>generally true</u>

Infintesimal Rotations

There is one other interesting point about rotations of 3-vectors ($\mathbb{R}_3$) $\rightarrow$ they can be done as a series of infintesimal steps from Identity.

Since the group of rotations is closed (by definition of group) we can imagine it as a sphere

- Of course, I need to go down from 3-D to 2 as a spherical surface is only 2-D.
- Imagine I can rotate about ~~x~~ z and y, but not x for some reason.

I can get from I to any point via a series of smooth infinitesimal steps.

In our case of rotation by α about z: $\alpha \rightarrow \delta$

$$R(\delta) = \begin{pmatrix} 1 & -\delta & 0 \\ +\delta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ to first order in δ

Which looks a lot like the first term in

$$\exp\left[\delta\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}\right] = 1+\delta\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix} \quad \text{Eqn. 1}$$

But didn't Noether's theroem say it should be
$R \propto \exp[L_z]$?

In fact, following our results from t-evolution, we can guess that it should probably be

$$R(\delta) = \exp\left[\frac{-i\delta}{\hbar}L_z\right] \sim 1 - \frac{i}{\hbar}\delta L_z$$

$$\sim 1 - \frac{i}{\hbar}\delta\begin{pmatrix}\hbar&0&0\\0&0&0\\0&0&-\hbar\end{pmatrix}$$

Or something?

While these don't look the same at first, I assert they are

Start by multiplying Eqn. 1 by $1 = \left(\frac{-i/\hbar}{-i/\hbar}\right)$

$$1 + \delta\cdot\left(\frac{-i}{\hbar}\right)\begin{pmatrix}0&-\hbar i&0\\+\hbar i&0&0\\0&0&0\end{pmatrix}$$

These two matricies only differ by a similarity transform

$$M^{-1}\begin{pmatrix}0&-\hbar i&0\\\hbar i&0&0\\0&0&0\end{pmatrix}M = \begin{pmatrix}\hbar&&\\&0&\\&&-\hbar\end{pmatrix} \quad \text{for } M = \begin{pmatrix}-i&0&i\\1&0&1\\0&1&0\end{pmatrix}$$

All we did is a change of basis! So $R_z = \exp\left[-\frac{i\alpha}{\hbar} L_z\right]$ holds.

What is the basis difference?

$$L_z = \begin{pmatrix} \hbar & & \\ & 0 & \\ & & -\hbar \end{pmatrix}$$ is in the spherical coordinates of the Y_{lm}. I.e. it acts on vectors of the type $\begin{pmatrix} Y_{11} \\ Y_{10} \\ Y_{1-1} \end{pmatrix}$

In contrast $L_z = \begin{pmatrix} 0 & -\hbar i & 0 \\ \hbar i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is in Cartesian coordinates

In fact you can see the connection in the transformation matrix M

$$M = \begin{pmatrix} -i & 0 & i \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$Y_{1-1} = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\, e^{-i\phi} \sin\theta = \frac{1}{2}\sqrt{\frac{3}{2\pi}}\, \frac{(x-iy)}{r}$$

$$Y_{10} = \frac{1}{2}\sqrt{\frac{3}{\pi}} \cos\theta = \frac{1}{2}\sqrt{3/\pi}\; z/r$$

$$Y_{11} = \frac{1}{2}\sqrt{3/2\pi}\, e^{i\phi} \sin\theta = \frac{1}{2}\sqrt{3/2\pi}\, \frac{(x+iy)}{r}$$

Slia

Lie-Groups

A Lie-Group is a continuous group where you can get to any group member via a series of infintesimal steps, just like we did here for rotations.

The Lie group is specified by a set of generators which have the same number as the number of free parameters in the group being generated.

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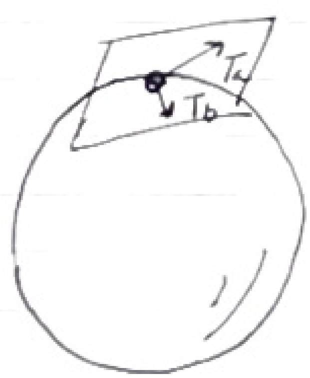
The group SO(3) has 3 parameters (the Euler angles) and 3 generators: L_z, L_x, and L_y
- Identity doesn't count as a generator

These generators, and their commutation relations, define a Lie algebra
- Generically $[T^a, T^b] = i f^{abc} T^c$
- In our case $[L_i, L_j] = i\hbar \epsilon_{ijk} L_k$
  - → $\hbar \epsilon_{ijk}$ are the structure constants for SO(3)

The Lie algebra for SO(3) is written $\mathfrak{so}(3)$

The generators act as unit vectors spanning the plane tangent to the group at identity



- Why a plane is bigger than a line even though both are infinite: more generators, more complex Lie algebra!

I can use these to get to any group element (just like we saw w/ Euler angles).

Why We used L=1

We were rotating vectors!
You may(?) know that the photon (a vector as we saw w/ parity) has spin=1, so things are at least logical!

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What about other values of angular momentum? 1/2 perhaps?

Spin 1/2 Objects need a 2-rep (not the 3 we've been working with)

Fortunately the Lie Algebra helps us out here

$$R(\varphi) = \exp\left[\frac{-i\varphi}{\hbar} \vec{S}\cdot\hat{n}\right]$$

where $\vec{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ for example and, in general, $S_i = \hbar/2\, \sigma_i$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Since these obey the same algebra, all good!

Or are we?

$$R(\varphi) = \exp\left[\frac{i}{\cancel{\hbar}}\varphi \frac{\cancel{\hbar}}{2} \vec{\sigma}\cdot\hat{n}\right] = \exp\left[i\left(\frac{\varphi}{2}\right)\vec{\sigma}\cdot\hat{n}\right]$$

I need to go 720° to get back to where I started!

Demo of Dirac belt trick.

SU(2) and Representations

The Pauli matricies are actually generators of $SU(2)$

- The group of all 2×2 unitary matricies of det = 1
- Also 3-parameters!

$$\left.\begin{array}{l} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow 4 \\ \rightarrow ad - bc = 1 \end{array}\right\} \begin{array}{l} (n^2 \text{ generally}) \\ 4-1=3 \end{array}$$

- I can make any $SU(2)$ matrix by $e^{-i\alpha_i \sigma^i}$

The fact that the number of parameters is the same for $SU(2)$ & $SO(3)$ is why they can have the same Lie algebra: $SU(2)$ is isomorphic to $SO(3)$

In fact locally, we can say that the rotation matricies we've been dealing with for vectors are the 3-rep of SU(2). The 2-rep is called the fundamental rep

Spin 3/2 would be in the 4-rep!

Generate the Lie Algebra of the correct dimensionality, and you can see how anything transforms under rotations!

Beyond that, nature really likes SU(2)

- Nuclear physics: p^+ & n^0 are almost a spin "up" & "down"
- The weak force is in a 3-rep of SU(2)
- ...