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Continuous Groups

AYACI

Summary of the definition of a group

- Closure, Identity, Inverse, Associativity
- Order - Number of elements
- Representations

Objectives

By the end of this class, you should be able to...

- ~~Describe how continuous groups can often be generated by a finite number of generators which themselves form a group.~~
- ~~Define a Lie group, and determine its dimensionality.~~
- Derive the generator of the group of time translations and show how this is connected to Noether's Theorem.
- Describe the differences between the Heisenberg and Schrödinger pictures of quantum mechanics.
- ~~Build a rotation matrix from its generators.~~
- ~~Use the correct basis of generators for a given ket basis.~~
- ~~Identify the generator of rotation as angular momentum.~~
- ~~Build representations of SU(2) for any dimensionality of vector space.~~
- ~~Explain the relationship between SU(2) and SO(3)~~

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Sources

For these notes, I am pulling from

- M. Peskin & D. Schroder. *An Introduction to Quantum Field Theory*. Westview Press, 1995. (pp 495 - 502).
- J. Mathews & R.L. Walker. *Mathematical Methods of Physics*, 2nd ed. Addison-Wesley, 1970 (pp. 449 - 466).
- My notes. Physics 234 A, University of California, Irvine. Prof. J. Feng. Winter 2009 Quarter.

Continuous Groups

Up to this point, we have been looking at discrete groups

- The D_3 of the triangle
- The D_8 of the square in your homework.
- Parity.

However, as you also saw in your homework, there are infinite continuous groups as well.

Eg. all $\mathbb{Z}$ <u>except</u> zero under multiplication.

Another, would be all $\mathbb{Z}$ with $|z| = 1$ under multiplication. Eg

$$z = e^{i\phi} \quad z_j = e^{i\phi_j} \quad z_k = e^{i\phi_k}$$

<u>closed</u>: $z_j z_k = \exp[i(\phi_j + \phi_k)]$

<u>identity</u>: $\phi = 0 \rightarrow z_I = 1$

<u>inverse</u>: $z_j^{-1} = \exp[-i\phi_j]$ and $z_j^{-1} z_j = I = 1$

<u>associative</u>: free by multiplication.

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Terminology

The group we just explored is called SU(1)

- 1 → 1-D matricies (i.e. numbers!)
- U → Unitary: $z^{+} = z^{-1}$
  - → Here, since 1-D $z^{+} = z^{*}$
- S → Special: $\det(z) = 1$
  - → Since 1-D $\det(z) = |z|$

An application of SU(1): Time-translation

In your first homework, you showed that in classical mechanics, a Lagrangian invariant under time translation $t \to t + \delta t$ ⇒ energy is conserved.

Now, we will look at it another way: energy is the reason time moves forward!

N.B. This is what I love about physics: these deep truths!

Let's define a time-translation operator T |

$T\Psi(x, t=0) = \Psi(x, t)$

Taylor expand $\Psi(x,t)$ about $t=0$:

$$\Psi(x,t) = \Psi(x,0) + \left.\frac{\partial \Psi}{\partial t}\right|_{t=0} t + \frac{1}{2!}\left.\frac{\partial^2 \Psi}{\partial t^2}\right|_{t=0} t^2 + \frac{1}{3!}\left.\frac{\partial^3 \Psi}{\partial t^3}\right|_{t=0} t^3 + \ldots$$

Factoring to the right

$$\Psi(x,t) = \left(1 + t\frac{\partial}{\partial t} + \frac{t^2}{2!}\frac{\partial^2}{\partial t^2} + \frac{t^3}{3!}\frac{\partial^3}{\partial t^3} + \dots\right)\Psi(x,0)$$

Multiply each term by a clever form of 1

$$1 = \left(\frac{i\hbar}{i\hbar}\right)^n$$ ← This b.t.w. is one of theoretical physics's favorite tricks! Don't forget it! Saved me many a time in grad school.

$$\Psi(x,t) = \left(1 + \frac{1}{i\hbar}t\left(i\hbar\frac{\partial}{\partial t}\right) + \frac{1}{(i\hbar)^2}\frac{t}{2!}\left((i\hbar)^2\frac{\partial^2}{\partial t^2}\right) + \dots\right)\Psi(x,0)$$

But, recall the Schrödinger Eqn:

$$H\Psi = i\hbar\frac{\partial}{\partial t}\Psi$$

$$\therefore \Psi(x,t) = \underbrace{\left(1 + \frac{1}{1!}\left(\frac{t}{i\hbar}\right)H + \frac{1}{2!}\left(\frac{t}{i\hbar}\right)^2 H^2 + \dots\right)}_{e^{-iHt/\hbar}}\Psi(x,0)$$

Recall $1/i = -i$

$$\therefore \Psi(x,t) = \exp\left[\frac{-iHt}{\hbar}\right]\Psi(x,0)$$

By correspondence $\mathcal{T} = e^{-iHt/\hbar}$!

The Hamiltonian is the cause (generator) of time translations!

We've now really come full circle on Noether's Theorem:

- Symmetry of time invariance → energy conservation
- Energy (in the form of the Hamiltonian) <u>causes</u> time to move forward!

Moreover, assuming H is a simple scalar (mostly true so far!) then the time translations are $\mathcal{U}(1)$!

<u>How thinking of H as the cause of time-evolution can change our understanding/view of QM</u>

How? Well, we know how to do a unitary transformation to an operator $\hat{\mathcal{O}}$:

$$\hat{\mathcal{O}}' = \mathcal{U}^{+}\hat{\mathcal{O}}\,\mathcal{U}$$

well the time-evolution operator is itself unitary, so, why not?

$$\hat{\mathcal{O}}(t) = \mathcal{T}^{+}(H,t)\,\mathcal{O}(t=0)\,\mathcal{T}(H,t)$$

It's hard to understate what a change this is: we now have our operators <u>changing in time</u>!

Up to now

We have thought of the time dependent wave-function as a linear combination of stationary states, each of fixed energy, with relative phases that are time dependent:

$$|\Psi\rangle = \sum_n c_n e^{-iE_nt/\hbar} |\psi_n\rangle$$

i.e. the wave-function itself evolves with time.

Say I want $\langle x\rangle$ (which in this case will be t-dependent)

$$\langle x\rangle = \langle\Psi|x|\Psi\rangle$$
$$= \left(\langle\psi_n|c_n^* e^{+iE_nt/\hbar}\right) x \left(c_n e^{-iE_nt/\hbar}|\psi_n\rangle\right)$$

(implied sum over n)

This is called the Schrödinger picture.

An alternative (aka Heisenberg picture)

Move the parentheses

$$\langle x\rangle = \left(\langle\psi_n|c_n^*\right) e^{+iE_nt/\hbar} x\, e^{-iE_nt/\hbar} \left(c_n|\psi_n\rangle\right)$$
$$= \left(\langle\psi_n|c_n^*\right) \mathcal{T}^\dagger(H,t)\, x\, \mathcal{T}(H,t) \left(c_n|\psi_n\rangle\right)$$
$$= \left(\langle\psi_n|c_n^*\right) x(H,t) \left(c_n|\psi_n\rangle\right)$$

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<u>Obviously the two are equivalent mathematically.</u>

Conceptually, however, there is a difference:

- The S-picture has a wave function that is extended across time and space <u>simultaneously</u>
  - → Potentially relativistically problematic!
- The H-picture avoids this as we can think of the operator's effect being localized with a time & space dependence.

Clock demo

Pronounced "Lee"

Rotations and More Complex Continuous (Lie) Groups

Objectives

By the end of this class, you should be able to...

- Show how angular momentum is the generator of rotations.
- Choose a suitable basis for the angular momentum operator for a given ket basis.
- Classify the group of rotations on vectors like $\vec{x}$ or $\vec{p}$.
- Define a Lie group, generators, Lie algebra, and structure constants.
- Determine the number of generators needed for a given Lie group.
- Define homomorphism and isomorphism.
- Explain why SO(3) & SU(2) are homomorphic.
- Generate representations of SU(2) of any dimensionality.

Sources: The sources of the last day's notes really apply here.

From Noether's Theorem, we can "guess" that if energy is the generator of time translations, then angular momentum will be the generator of rotations.

$$R(\varphi) \propto e^{L_n} \leftarrow \text{Keep}$$

where $R_{\hat{n}}(\varphi)$ is a rotation about the $\hat{n}$-axis by φ.

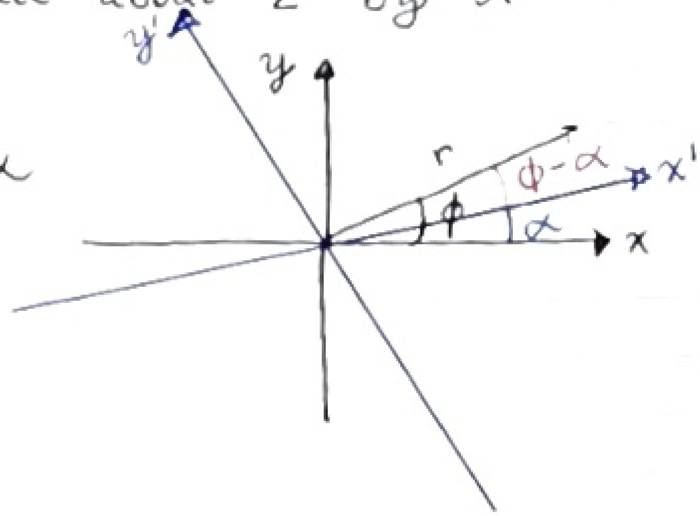
Since the choice of $\hat{z}$ is arbitrary, we may as well $\hat{z} = \hat{n}$

Thus, let's rotate about z by α:

$r \to r$

$\theta \to \theta$

$\phi \to \phi - \alpha$



Slide to animate & keep simpler and ⊥

N.B. we are implicitly rotating a 3-vector here (really only 2) as that is the most familiar!

$x' = x\cos\alpha - y\sin\alpha$

$y' = x\sin\alpha + y\cos\alpha$

$$\therefore \quad R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is a matrix that is
- Real
- Orthogonal: $R^T(\varphi)R(\varphi) = 1$ (rotating back by φ gets me back to where I started)
- determinant $= 1$

Of course, this is only a rotation about $\hat{z}$ by φ

To do <u>any</u> generic rotation I need 3 angles
- The Euler angles are common
- We did the first
- Then, rotate about y' by β
- Then, about z' by γ

} Slide w/ diagram

Thus, any rotation becomes

$$\mathcal{R}(\alpha,\beta,\gamma) = R_{z'}(\gamma)\, R_{y'}(\beta)\, R_z(\alpha)$$

- $R_{y'}(\beta) = R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha)$
  - → Undo the z rotation bringing $y' = y$
  - → Rotate about $y = y'$ by β
  - → Re-rotate z
- $R_{z'}(\gamma) = R_{y'}(\beta) R_z(\gamma) R_{y'}^{-1}(\beta)$
- Put all back together → $R(\alpha,\beta,\gamma) = R_z(\alpha) R_y(\beta) R_z(\gamma)$

$\boxed{\text{Key Takeaway:}}$ You can describe any rotation with a 3x3 real, orthogonal matrix!

Rotations of vectors are SO(3)

1) Any rotation can be represented by a 3x3 matrix and the combination of any two rotations is itself a rotation (see the Euler angles above) → **closed ✓**
2) I can do nothing → **Identity represented by $I_{3\times3}$ ✓**
3) There is an inverse (again see discussion of Euler angles above) → **Inverse ✓**
4) We have already been in a representation of 3x3 matricies and matrix multiplication is associative
   ↳ **Associativity ✓**

What is the dimensionality?

For discrete groups, dim(G) was simply the number of elements.

But there are an infinity of SO(3) matricies

Or are there? How many independent components?

$$R = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$ Looks like $n^2 = 9$, but

$$R^T R = 1_{3\times 3} \Rightarrow \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

$\hookrightarrow$

$$a^2 + d^2 + g^2 = 1 \qquad ab + de + gh = 0$$
$$b^2 + e^2 + h^2 = 1 \qquad bc + ef + hi = 0$$
$$c^2 + f^2 + i^2 = 1 \qquad ac + df + gi = 0$$

9 parameters − 6 eqn = 3 d.o.f. which we already knew

- α, β, γ Euler angles
- Pitch, roll, yaw

In general $\dim(SO(N)) = \dfrac{N(N-1)}{2}$

This dimensionality as number of free parameters is <u>generally true</u>