Need: Triangles

N.B.: Terms are in *cursive* to highlight them.

8Feb

Start Group Theory

A.Y.A.C.I.

Einsteins Obit of Noether & Get a triangle

Slide

Objectives

By the end of today, you should be able to...

- List the criteria of a mathematical group.
- Define: *closure*, *identity*, *inverse*, and *associativity* in the context of groups. Also *multiplication table*.
- Define *representation* in a group theory context.
- Build a multiplication table.
- From a multiplication table, determine if a group is Albelian or not.
- Build the D_3 multiplication table
- Determine representations of various dimensionality.
- Determine if a set is a group or not.

Slide

Summary of Last Time

Getting More Formal With Symmetries via Equalateral Triangles as a Touchstone

You have a triangle labelled I, II, III.

N.B. The reverse is equally labelled.

<u>N.B.</u> A <u>lot</u> of terms today! Get them all!

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Grou

What operators can we apply to the triangle which leave it invariant ("pointing up")? Ex

- $R_+ \rightarrow$ Rotate CCW $120°$
- $R_- \rightarrow$ Rotate CW $120°$
- Do nothing : I
- $R_A \rightarrow$ Flip about I
- $R_B \rightarrow$ Flip about II
- $R_C \rightarrow$ Flip about III

Using $R_{A,B,C}$ as R_{III} is gross!

What about a CCW $240°$ rotation? Ex

- That is just a $120°$ CW!
- $R_+ R_+ = R_+^2 = R_-$

These are it! We have a complete **Group** of operators.

- I use this word intentionally, we are studying a branch of mathematics called **group theory**
- Group theory is a <u>key</u> piece of modern physics
  - → Just as linear alg drove the development of QM, group theory is a big driver in modern theor. physics.
  - → The SM, for example, is <u>defined</u> by its group theory structure $SU(3) \times SU(2) \times U(1)$
- This group has 6 operators, so we call it **order** 6

General Properties of Groups

1) A set of **elements**. $\{g_i\} \in G$
   - Here, operations to our triangles
2) An operation we call **multiplication**. ← Here, ope... in seque...
   - Not ~~really~~ necessarily ~~ab~~ $2 \cdot 3 = 6$
   - Need not be communicative!
3) Must have the following properties

   (1) **Closure** – The result of our operations (any number, in any order) must also be in the group.

   (2) **Identity** – $\exists$ an operation I | $I g_i = g_i I = g_i$ for any $g_i \in G$

   (3) **Inverse** – $\forall g_i \; \exists \; g_i^{-1} \mid g_i g_i^{-1} = g_i^{-1} g_i = I$

   (4) **Associativity** – $(g_i g_j) g_k = g_i (g_j g_k)$

Does this hold for our set of triangle operators?

1) Closure? ✓ (By construction!)
2) Identity? ✓ (Do nothing!)
3) Inverse? ✓
   - $R_+^{-1} = R_-$
   - $R_A^{-1} = R_A$!

Also see our $R_+^2 = R_-$

Ex

4) Associativity? ✓
   - $R_+(R_A R_-) = (R_+ R_A) R_-$

Are these groups? Ex
1) Integers under addition? Y
2) Integers under multiplication? N. Need fractions

Multiplication Tables

Any group can be defined by its **multiplication table**.

First→ / Second↓	I	R_+	R_-	R_A	R_B	R_C
I	I	R_+	R_-	R_A	R_B	R_C
R_+	R_+	R_-	I	R_B	R_C	R_A
R_-	R_-	I	R_+	R_C	R_A	R_B
R_A	R_A	R_B	R_C	I	R_+	R_-
R_B	R_B	R_C	R_A	R_-	I	R_+
R_C	R_C	R_A	R_B	R_+	R_-	I

Ex

Do these operators commute? Ex
- No.
- Non-Albelian
  - → Actually, this is the smallest non-Albelian group FYI!

But there is an order 3 **subgroup!** which is Albelian

☆☆ ANYTHING THAT HAS THIS SAME MULTIPLICATION TABLE IS THE SAME GROUP ☆☆
↳ There are multiple **representations** of every group.

Representations

Still working in the context of our triangle example.

The simplest to figure out is 3x3 Matricies

Eg

R_+ :

$$R_+ \begin{pmatrix} \text{top} \\ \text{lower right} \\ \text{lower left} \end{pmatrix}$$

$$R_+ \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} B \\ C \\ A \end{pmatrix} \rightarrow R_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Y'all do the rest

Ex

$$R_- = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad R_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$R_B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad R_C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

dim: Dimensionality

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This is all possible 3×3 matricies w/ a single 1 in each row or column N.B.

- In addition to all operations on a triangle, this group also represents all possible ways to arrange three objects!
- ↳ the name: D_3 (Don't need to know!)

There are others

Represent Everything w/ 1's

Still meets the table (if stupidly!)

Called the Trivial Representation

Another 1-D Representation

$I = R_+ = R_- = 1$

$R_A = R_B = R_C = -1$

This is a 1-Rep : all elements have dim. 1

2-Rep

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$R_+ = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \qquad R_- = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$R_A = \begin{pmatrix} \frac{1}{2} & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \qquad R_B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad R_C = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$