Angular Momentum Review[1]

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Notation, Conventions, Summary and Toolkit

- The canonical angular momentum commutation relation is

$$[L_i, L_j] = i\hbar\epsilon_{ijk}L_k$$

- For a spin-one-half (spin-1/2) system, the spin quantum number $s = 1/2$ and the quantum number for the z-component of the spin is $m_z = \pm 1/2$. Therefore, if we choose the orthonormal basis vectors to be the eigenstates of the operators S^2 and S_z, we can denote them as $|s, m_z\rangle$:

$$\left|\frac{1}{2}, +\frac{1}{2}\right\rangle \text{ and } \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$$

- Since the spin quantum number $s = 1/2$ is fixed, we can simplify the notation for the basis vectors as

$$|s, m_z\rangle = |m_z\rangle = \left|\pm\frac{1}{2}\right\rangle$$

- Another common notation for the basis vectors $|s, m_z\rangle$ for a spin-1/2 system is
  - $\left|+\frac{1}{2}\right\rangle = |\uparrow\rangle$
  - $\left|-\frac{1}{2}\right\rangle = |\downarrow\rangle$
- The eigenvalue equations for the operators S_z and S^2 are given by:
  - $S_z|\uparrow\rangle = \frac{\hbar}{2}|\uparrow\rangle$ and $S_z|\downarrow\rangle = -\frac{\hbar}{2}|\downarrow\rangle$
  - $S^2|\uparrow\rangle = \hbar^2 s(s+1)|\uparrow\rangle \xrightarrow{s=1/2} S^2|\uparrow\rangle = \frac{3\hbar^2}{4}|\uparrow\rangle$
  - $S^2|\downarrow\rangle = \hbar^2 s(s+1)|\downarrow\rangle \xrightarrow{s=\frac{1}{2}} S^2|\downarrow\rangle = \frac{3\hbar^2}{4}|\downarrow\rangle$
- In the algebra of angular momentum, the raising and lowering operators have the following effect on the simultaneous eigenstates of S^2 and S_z:

$$S_\pm|s, m_z\rangle = \hbar\sqrt{s(s+1) - m_z(m_z \pm 1)}\,|s, (m_z \pm 1)\rangle$$

  - The raising operator acting on the highest m_z state annihilates it.
  - The raising operator acting on the lowest m_z state annihilates it.
  - $S_\pm = S_x \pm iS_y$

[1] This tutorial is based on the Product Space Warmup Pretest and Spin Concept Tests 1-3 from the QUILTs: https://www.physport.org/curricula/quilts/

Questions

For a spin-1/2 system, the spin quantum number $s = 1/2$ and the quantum number for the z component of the spin is $m_z = \pm 1/2$. The matrix representations for $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$, which are the eigenstates of $\hat{S}_z$, are $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$, respectively. Choose all of the following statements that are true.

(I) $\left|\frac{1}{2},\frac{1}{2}\right\rangle = |1/2\rangle = |\uparrow\rangle_z$

(II) $\left|-\frac{1}{2},-\frac{1}{2}\right\rangle = |-1/2\rangle = |\downarrow\rangle_z$

(III) $\hat{S}_z|\downarrow\rangle_z = \hbar/2|\downarrow\rangle_z$

This is just a representation question to make sure folks understand the notation. I and III are true. Choice II is not true as the first number refers to S which is always +1/2.

The matrix representation for $|\uparrow\rangle_z$, which is an eigenstate of $\hat{S}_z$, is $\begin{pmatrix}1\\0\end{pmatrix}$. Which of the following statements are correct about $|\uparrow\rangle_z$?

(I) $|\uparrow\rangle_z = \frac{1}{\sqrt{2}}(|\uparrow\rangle_x + |\downarrow\rangle_x)$

(II) $|\uparrow\rangle_z = \frac{1}{\sqrt{2}}(|\uparrow\rangle_x - |\downarrow\rangle_x)$

(III) $|\uparrow\rangle_z = \frac{1}{\sqrt{2}}(|\uparrow\rangle_y - |\downarrow\rangle_y)$

This one we did in class. We saw that I and III are true.

At time $t=0$, the initial state of a spin-1/2 particle is $|\uparrow\rangle_z$. Choose all of the following statements that are correct.

(1) If we perform a measurement of the x or y component of the spin at $t=0$, we will obtain zero for both.

(2) If we measure $\hat{S}_y$, we can get either $\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$ with equal probability.

(3) If we measure $\hat{S}^2$, we can only get $\frac{3\hbar^2}{4}$.

(3) is manifestly true. From

$$|\uparrow\rangle_y = \frac{1}{\sqrt{2}}\left[\,|\uparrow\rangle_z + i\,|\downarrow\rangle_z\right]$$

And

$$|\downarrow\rangle_y = \frac{1}{\sqrt{2}}\left[\,|\uparrow\rangle_z - i\,|\downarrow\rangle_z\right]$$

We see that

$$|\uparrow\rangle_z = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_y + |\downarrow\rangle_y\right]$$

Which means that we can get $\pm\hbar/2$ with equal probability, so (2) is also true. The fact that (2) is true, implies that (1) is false.

Suppose the initial spin state of an electron is $a|\uparrow\rangle_z + b|\downarrow\rangle_z$ where a and b are non-zero constants. Choose all of the following statements that are correct.

(1) If we measure the z-component of the spin of the electron, we can obtain $+\hbar/2$ with a probability $|a|^2$.

(2) If we measure the x-component of the spin of the electron, we can obtain $+\hbar/2$ with a probability $|a|^2$.

(3) If we measure the x-component of the spin of the electron, we can obtain $+\hbar/2$ with a probability $\frac{1}{2}|a+b|^2$.

Choice (1) is fairly obviously true. Choice (3) is also true, which precludes choice (2) (see page 169 in Griffiths for this!).

Suppose the particle is initially in an eigenstate of the x-component of spin angular momentum operator $\hat{S}_x$. Choose all of the following statements that are correct:

(1) The expectation value $\langle S_x\rangle$ depends on time.

(2) The expectation value $\langle S_y\rangle$ depends on time.

(3) The expectation value $\langle S_z\rangle$ depends on time.

None of these are dependent on time as the probabilities are simply fixed. This will be true unless the different spins have different energies.

Suppose the particle is initially in an eigenstate of the z-component of spin angular momentum $\hat{S}_z$. Choose all of the following statements that are correct:

(1) The expectation value $\langle S_x \rangle$ depends on time.

(2) The expectation value $\langle S_y \rangle$ depends on time.

(3) The expectation value $\langle S_z \rangle$ depends on time.

Same as above.

Choose all of the following statements that are true about the Hamiltonian $H_0 = \text{constant } S_z$ for a spin-1/2 system:

1. If we choose two different bases (coordinates), H_0 may be a diagonal matrix in one basis but not in the other.
2. If the basis vectors are eigenstates of H_0, H_0 will be a diagonal matrix.
3. No matter what basis we choose, H_0 must always be a diagonal matrix by definition.

Choices (1) and (2) are generally true for any quantum mechanical system.

What is the matrix representation of S^2 if we choose the basis vectors in the order $\{|\uparrow\rangle, |\downarrow\rangle\}$ to construct the matrix.

$$S^2 = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

What is the correct expression for $\langle\downarrow|S_-|\uparrow\rangle$ for a spin-1/2 system?

$$S_\pm|s\ m\rangle = \sqrt{s(s+1)-m(m\pm 1)}\ |s\ m\pm 1\rangle \quad \text{(Eqn. 4.136 in Griffiths)}$$

In our case

$$S_-|\uparrow\rangle = \sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}-1\right)}\ |\downarrow\rangle$$

$$S_-|\uparrow\rangle = \sqrt{\frac{1}{2}\left(\frac{3}{2}\right)-\frac{1}{2}\left(-\frac{1}{2}\right)}\ |\downarrow\rangle$$

$$S_-|\uparrow\rangle = \sqrt{\frac{3}{4}+\frac{1}{4}}\ |\downarrow\rangle$$

$$\langle\downarrow|S_-|\uparrow\rangle = \langle\downarrow|(1\ |\downarrow\rangle) = 1$$

Consider a single spin-1/2 system (*e.g.*, an electron spin). What is the dimensionality of the vector space associated with the spin degrees of freedom?

(a) 2-dimensional
(b) 3-dimensional
(c) 4-dimensional
(d) finite dimensional but can be anything
(e) infinite dimensional

Consider a single spin-1 system (*e.g.*, spin of the W boson). What is the dimensionality of the vector space associated with the spin degrees of freedom?

(a) 2-dimensional
(b) 3-dimensional
(c) 4-dimensional
(d) finite dimensional but can be anything
(e) infinite dimensional

Consider a single spin-3/2 system. What is the dimensionality of the vector space associated with the spin degrees of freedom?

(a) 2-dimensional
(b) 3-dimensional
(c) 4-dimensional
(d) finite dimensional but can be anything
(e) infinite dimensional

Write S_z for a spin-3/2 system.

$$S_z = \hbar \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}$$

Crystals as application of the translation operator

a) Write the operator which translates $\psi(x) \to \psi(x+a)$

Noether's theorem suggests that the generator of translation is momentum. In this case, it would be the crystal momentum q:

$$T(a) = \exp\left[\frac{iqa}{\hbar}\right]$$

In rior examples, the argument of the exponential carried a negative sign, but that was because we translated the "other way" $\psi(x) \to \psi(x-a)$

b) The periodic boundry conditions yield a quantization condition. What is it?

The periodic b.c.'s mean that

$$\psi(x+Na) = \psi(x)$$

Which, in terms of our translation operator is

$$\mathcal{T}(Na)\psi(x) = \psi(x)$$

$$e^{iqNa/\hbar}\psi(x) = \psi(x)$$

meaning $\frac{iqNa}{\hbar} = 1 \rightarrow \frac{qNa}{\hbar} = 2\pi n$

$$\therefore \quad q = \frac{2\pi n\hbar}{Na}$$