# Homework #3

1A

1. Figure out the space translation operator $\mathcal{T}(a)$ such that

$$\mathcal{T}(a)\,\psi(x) = \psi(x-a)$$

Taylor expand $\psi(x-a)$ about $x=0$*

$$\psi(x-a) = \psi(0) + \left.\frac{\partial\psi}{\partial x}\right|_{x=0}(0-a) + \frac{1}{2!}\left.\frac{\partial^2\psi}{\partial x^2}\right|_{x=0}(0-a)^2 + \dots$$

Factoring to the right as we did in class:

$$\psi(x-a) = \left[1 - a\cdot\frac{\partial}{\partial x} + \frac{(-a)^2}{2!}\frac{\partial^2}{\partial x^2} + \dots\right]\psi$$

$[a, \partial/\partial x] = 0$ because $a$ is a number.

Multiply by $1 = \left(\frac{-i\hbar}{-i\hbar}\right)^n$

$$\psi(x-a) = \left[1 + \left(\frac{-a}{-i\hbar}\right)(-i\hbar)\frac{\partial}{\partial x} + \frac{1}{2!}\left(\frac{-a}{-i\hbar}\right)^2\left(-i\hbar\frac{\partial}{\partial x}\right)^2 + \dots\right]\psi$$

$$-i\hbar\frac{\partial}{\partial x} = \hat{p}$$

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* Yes, this implies a radius of convergence from zero to a. However, if $a \to 0$ this is ok and we get the generator of infintesimal translations. If the convergence <u>is</u> there, it works for finite.

$$\Psi(x-a) = \left[1 + \left(\frac{-a}{-i\hbar}\right)P + \frac{1}{2!}\left(\frac{-a}{-i\hbar}\right)^2 P^2 + \cdots\right]\Psi$$

$$\Psi(x-a) = \left[1 + \left(\frac{-ia}{\hbar}\right)P + \frac{1}{2!}\left(\frac{-ia}{\hbar}\right)^2 P^2 + \cdots\right]\Psi$$

$$\Psi(x-a) = \exp\left[\frac{-iaP}{\hbar}\right]\Psi$$

$\therefore$ By correspondance

$\mathcal{P}(a) = \exp\left[\frac{-ia\hat{P}}{\hbar}\right]$ → Momentum is generator of translation as we saw in-class!

Which, if you look at eqn 6.3 in your book, is correct. Moreover, it does match what we would expect from our other generators

$$\mathcal{O}(\text{parameter}) = \exp\left[\frac{-i}{\hbar}(\text{parameter})(\text{generator})\right]$$

2. (Griffiths 6.18) Considering a free particle in 1-D:

$$H = \frac{p^2}{2m}$$

which has both translation & inversion symmetry

a) Show that translations & inversions don't commute.

Basically, we are looking at the commutation between the parity operator $\Pi$ (in 1-D $\Pi$ and inversion are the same), and the translation operator $\mathcal{P}$ we saw in the last problem:

$$[\Pi, \mathcal{P}] \neq 0$$

As usual, the best approach is to act this on an arbitrary wave function $\psi(x)$.

$$[\Pi, \mathcal{P}]\psi = \Pi\mathcal{P}\psi - \mathcal{P}\Pi\psi$$

$\mathcal{P}\psi(x) = \psi(x-a)$ by definition. Similarly, $\Pi\psi(x) = \psi(-x)$.

$$[\Pi, \mathcal{P}]\psi(x) = \Pi\psi(x-a) - \mathcal{P}\psi(-x)$$

$$= \psi(-x-a) - \psi(-(x-a))$$

$$= \psi(-x-a) - \psi(-x+a) \neq 0 \quad \text{Q.E.D.}$$

b) Because of the translational symmetry, eigenstates of H are also eigenstates of p. Show that the parity operator flips the sign of p.

The eigenfunctions for the free particle are

$$\psi(x,p) = \exp[ipx/\hbar]$$

$$\Pi\psi = \exp[ip(-x)/\hbar]$$

Simply moving the negative sign yields

$$\Pi\psi = \exp[i(-p)x/\hbar] = \psi(x,-p)$$

c) Because of parity, we know that the eigenstates of H can also be written in parity eigenstates, i.e.

$$\psi_+(x,p) = \frac{1}{\sqrt{\pi\hbar}}\cos\left(\frac{px}{\hbar}\right) \qquad \psi_-(x,p) = \frac{1}{\sqrt{\pi\hbar}}\sin\left(\frac{px}{\hbar}\right)$$

Show that translation mixes these states (therefore, they <u>must</u> be degenerate).

$$\mathcal{D}\psi_+(x,p) = \frac{1}{\sqrt{\pi\hbar}}\cos\left(\frac{p(x-a)}{\hbar}\right)$$

By $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$, P564

2C

$$\mathcal{P}\Psi_+(x,p) = \frac{1}{\sqrt{\pi\hbar}}\left[\cos\left(\frac{pa}{\hbar}\right)\cos\left(\frac{px}{\hbar}\right) + \sin\left(\frac{pa}{\hbar}\right)\sin\left(\frac{px}{\hbar}\right)\right]$$

This is the odd parity eigenstate $\Psi_-$
mixed in.

For the last two problems, I am doing it in Mathematica as there is a lot of brute force.

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# 3. Orthogonality of generators

## a) Show that the 2-rep $su(2)$ Pauli matrices satisfy the orthogonality condition $\mathrm{tr}\left[T_r^a\, T_r^b\right] = C(r)\, \delta_{ab}$

In[ ]:=
```
σx = {{0 , 1}, {1, 0}}; MatrixForm[σx]
```

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In[ ]:=
```
σy = {{0, -ⅈ}, {ⅈ, 0}}; MatrixForm[σy]
```

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

In[ ]:=
```
σz = {{1, 0}, {0, -1}}; MatrixForm[σz]
```

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

First, lets calculate the traces that mix $\sigma_i$'s. Because of the $\delta_{ab}$, these should all be zero.

In[ ]:=
```
Tr[σx.σy]
```

Out[ ]=

0

In[ ]:=
```
Tr[σx.σz]
```

Out[ ]=

0

In[ ]:=
```
Tr[σy.σz]
```

Out[ ]=

0

Now, let's do the ones that are $\sigma_i^2$

In[ ]:=
```
Tr[σx.σx]
```

Out[ ]=

2

In[ ]:=
```
Tr[σy.σy]
```

Out[ ]=

2

In[ ]:=
```
Tr[σz.σz]
```

Out[ ]=

2

Thus, we see that tr$\left[T_r^a\, T_r^b\right] = C(r)\, \delta_{ab}$ with $C(r) = 2$

### b) Show that the matrix $D_2^{a\,b} \equiv$ tr$\left[T_2^a\, T_2^b\right]$ is positive definite.

Complex positive definite matrices are Hermitian and have real eigenvalues. From above, we can see that tr$\left[T_2^a\, T_2^b\right]$ is

```
In[∘]:= Dab = {{2, 0, 0}, {0, 2, 0}, {0, 0, 2}}; MatrixForm[Dab]
```

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

This is manifestly Hermitian. The eigenvalues are also all positive

```
In[∘]:= Eigenvalues[Dab]
Out[∘]=
{2, 2, 2}
```

Q.E.D.

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# 3. The 4-rep of $s\,u\,(2)$.

### a) What are the generators $T_4^j$? How many generators must there be? In what basis did you write them?

This is still SU(2), thus, there are still three generators. The easiest one to write is $T_4^z$ which will just be the possible values on the diagonal. A 4-representation will be $S = \frac{3}{2}$, so

```
In[∘]:= T4z = {{3/2, 0, 0, 0}, {0, 1/2, 0, 0}, {0, 0, -1/2, 0}, {0, 0, 0, 3/2}};
MatrixForm[T4z]
```

$$\begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{3}{2} \end{pmatrix}$$

Which is the same as the $S_z\left(s = \frac{3}{2}\right)$ matrix. This is in the spherical harmonic $Y_l^m$ basis.

Now, all we need are the $T_4^x$ and $T_4^y$. To find, these, we can create the raising and lowering operators $S_+$ and $S_-$ because $S_\pm = S_x + i\, S_y$ implying that $S_x = \frac{1}{2}\,(S_+ + S_-)$ and $S_y = \frac{1}{2\,i}\,(S_+ - S_-)$. (See page 168 in your text for details).

$S_{\pm}$, by definition, have the property $S_{\pm} \left| s\, m \right\rangle = \sqrt{s\,(s+1) - m\,(m \pm 1)} \left| s\,(m \pm 1) \right\rangle$ which means that

```
In[ ]:= Splus = {{0, Sqrt[3], 0, 0}, {0, 0, 2, 0}, {0, 0, 0, Sqrt[3]}, {0, 0, 0, 0}};
MatrixForm[Splus]
```

$$\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

```
In[ ]:= Sminus = {{0, 0, 0, 0}, {Sqrt[3], 0, 0, 0}, {0, 2, 0, 0}, {0, 0, Sqrt[3], 0}};
MatrixForm[Sminus]
```

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

Giving us

```
In[ ]:= T4x = 1/2 {{0, Sqrt[3], 0, 0}, {Sqrt[3], 0, 2, 0},
   {0, 2, 0, Sqrt[3]}, {0, 0, Sqrt[3], 0}}; MatrixForm[T4x]
```

$$\begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}$$

and

```
In[ ]:= T4y = 1/(2 i) {{0, Sqrt[3], 0, 0}, {-Sqrt[3], 0, 2, 0},
   {0, -2, 0, Sqrt[3]}, {0, 0, -Sqrt[3], 0}}; MatrixForm[T4y]
```

$$\begin{pmatrix} 0 & -\frac{i\sqrt{3}}{2} & 0 & 0 \\ \frac{i\sqrt{3}}{2} & 0 & -i & 0 \\ 0 & i & 0 & -\frac{i\sqrt{3}}{2} \\ 0 & 0 & \frac{i\sqrt{3}}{2} & 0 \end{pmatrix}$$

## b) Determine the Casimir operator $C(4)$ for $su(2)$.

We have already shown that, in the 2-rep case, the $C(r)$ is a constant for a given representation. So we really only need to do one.

In[ ]:=
```
C4 = Tr[T4z.T4z]
```
Out[ ]=
5

Now, there are alternative values depending on how you normalized your matrices (if you pulled a 3/2 out front to match the Pauli matrices in question 1 for example).