each page
in case they
come unstapled.

P564 SP'24

Number this way,
easy to know which
problem

1A

Homework 1

NB. For this first solution set, I will make some notes on best practices etc. that will help you get the full credit for writing quality as well as just general good habits. These are in red.

1. Using $\langle x^2 \rangle$ and $\langle p^2 \rangle$, prove the virial theorem in the case of the QMHO.

$$\langle V \rangle = \langle T \rangle = \langle H \rangle / 2$$

It is good practice to write the problem, at least in summary. It makes it easier to follow & study later. I use blue for this to make it stand out.

We'll start with $\langle x^2 \rangle$:

$$\langle x^2 \rangle = \left\langle n \left| \frac{\hbar}{2m\omega}\left(a_-^2 + a_-a_+ + a_+a_- + a_+^2\right) \right| n \right\rangle$$

The squared terms will always go to zero by orthogonality.

Note this explanation. Also note that I don't use more than 1 = sign per line. Equals means "equals" NOT "and then".

HW #1 1B

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\langle n|(a_-a_+ + a_+a_-)|n\rangle$$

Recalling that

$$a_+|n\rangle = \sqrt{n+1}\,|n\rangle \quad \text{and} \quad a_-|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\left(\sqrt{n+1}\sqrt{n+1} + \sqrt{n}\sqrt{n}\right)$$

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\left(n + (n+1)\right)$$

The $\langle p^2\rangle$ follows similarly:

$$\langle p^2\rangle = \left\langle n\right| -\frac{m\hbar\omega}{2}\left(a_+^2 - a_-a_+ - a_+a_- + a_-^2\right)\left|n\right\rangle$$

Again, all squared terms are zero as before

$$\langle p^2\rangle = +\frac{m\hbar\omega}{2}\langle n|a_-a_+ + a_+a_-|n\rangle$$

$$= \frac{m\hbar\omega}{2}\left(n + (n+1)\right)$$

Now, for the virial theorem, we need $\langle V\rangle$

Use an arrow for multiple steps on one line.

$$\langle V\rangle = \left\langle \frac{1}{2}m\omega^2 x^2\right\rangle \rightarrow \langle V\rangle = \frac{1}{2}m\omega^2\langle x^2\rangle$$

HW #1 1C

$$\langle V \rangle = \frac{1}{2}\cancel{m}\omega^{\cancel{2}}\left[\frac{\hbar}{2\cancel{m}\cancel{\alpha}}(2n+1)\right]$$

$$\langle V \rangle = \frac{\hbar\omega}{4}(2n+1)$$

Pulling out the 2 from the $(2n+1)$
$\rightarrow 2(n+\frac{1}{2})$

$$\langle V \rangle = \frac{\hbar\omega \cdot 2}{4}(n+\frac{1}{2})$$

$$\langle V \rangle = \frac{\hbar\omega}{2}(n+\frac{1}{2}) = \frac{\langle H \rangle}{2}$$

← Here, double equal are appropriate as my goal is to say the middle is equal to both.

Once we recall
$\langle H \rangle = \hbar\omega(n+\frac{1}{2}) = E_n$

We, of course, also need $\langle T \rangle$

$$\langle T \rangle = \left\langle \frac{p^2}{2m} \right\rangle$$

$$\langle T \rangle = \frac{1}{2m}\langle p^2 \rangle$$

$$\langle T \rangle = \frac{1}{2\cancel{m}}\left[\frac{\cancel{m}\hbar\omega}{2}(2n+1)\right]$$

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HW #1

1D

$$\langle T \rangle = \frac{\hbar\omega}{4}(2n+1)$$

$$\langle T \rangle = \frac{1}{2}\hbar\omega\left(n+\tfrac{1}{2}\right) = \frac{\langle H \rangle}{2}$$

$$\therefore \quad \langle V \rangle = \langle T \rangle = \frac{\langle H \rangle}{2} \quad \text{Q.E.D.}$$

HW #1 Starting each on a new page $\Rightarrow$ 2A

2. Show that, for $L(x_i, \dot{x}_i)$ $(\partial L/\partial t = 0)$, a symmetry w.r.t. t $(t \to \delta t)$ implies conservation of energy. You will need to assume $\partial U/\partial t = \partial U/\partial \dot{x} = 0$.

$$\frac{dL}{dt} = \frac{\partial L}{\partial x_i}\dot{x}_i + \frac{\partial L}{\partial \dot{x}_i}\ddot{x}_i + \cancelto{0}{\frac{\partial L}{\partial t}}$$ by assumption

Recall the Euler-Lagrange equations

$$\frac{\partial L}{\partial x_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}$$

which, after substitution, yields

$$\frac{dL}{dt} = \dot{x}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} + \frac{\partial L}{\partial \dot{x}_i}\ddot{x}_i$$

N.B. I'm using the Einstein convention of repeated incicies imply summation.

We can "un-chain-rule" to get

$$\frac{dL}{dt} = \frac{d}{dt}\left(\dot{x}_i \frac{\partial L}{\partial \dot{x}_i}\right)$$

$$\frac{dL}{dt} - \frac{d}{dt}\left(\dot{x}_i \frac{\partial L}{\partial \dot{x}_i}\right) = 0$$

HW #1

Pulling the time derivative outside:

$$\frac{d}{dt}\left(L - \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} \right) = 0$$

which implies

$$L - \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} = \text{constant}$$

Now, $L \equiv T - U$ so

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} - \cancelto{0}{\frac{\partial U}{\partial \dot{x}_i}}$$ 0 by assumptions of the problem

which leaves us with

Eqn numbers are useful. Use them!

$$L - \dot{x}_i \frac{\partial T}{\partial \dot{x}_i} = \text{Constant} \qquad \text{(Eqn. 1)}$$

Euler's theorem says that for any homogeneous function $f(y_k)$

$$y_k \frac{\partial f}{\partial y_k} = nf$$

where n is the degree of the function. In our case, the degree of T is $n=2$ ($T = \frac{1}{2} m \dot{x}_i^2$ after all!)

HW #1 2C

Thus, by Euler's Theorem

$$\dot{x}_i \frac{\partial T}{\partial \dot{x}_i} = 2T$$

Which, when substituted into Eqn. 1 along with $L = T - U$ yields

$$(T-U) - 2T = \text{Constant}$$

$$-T - U = \text{Constant}$$

$$T + U = \text{Constant}$$

Where we recognize the constant as the total energy.

Q.E.D.

HW #1 3A

3. Which of the following are groups

a) All real numbers under multiplication

This is NOT a group. The issue is inverse: the element zero does not have a real inverse which yields 1. I.e.

$$\nexists \; g_i \in \mathbb{R} \;|\; g_i \cdot 0 = \underline{1}$$

b) Real numbers under addition

Closure: $\forall \{g_i, g_k\} \in \mathbb{R}, g_i + g_k \in \mathbb{R}$ ✓

Identity: Zero, which is in $\mathbb{R}$, is the identity: $\forall g_i \in \mathbb{R}, 0 + g_i = g_i$ (including when $g_i = 0$!) ✓

Inverse: $\forall g_i \in \mathbb{R} \; \exists \; g_i^{-1} = -g_i$ which is also in $\mathbb{R}$. Even zero works: $0^{-1} = -0, -0 + 0 = 0$ ✓

Associativity: $(a+b)+c = a+(b+c)$ ✓

HW #1 3B

c) All complex ($\mathbb{C}$) numbers, except zero, under usual multiplication

Closure: If $\{z_i, z_j\} \neq 0$ and $\{z_i, z_j\} \in \mathbb{C}$, then $z_i z_j \in \mathbb{C}$ and $\neq 0$. ✓

Identity: $I = 1$ $\forall z_i \in \mathbb{C}$, $1 \cdot z_i = z_i$ ✓

Inverse: $\forall z_i \in \mathbb{C}$ $\exists$ $z_i^{-1} = 1/z_i \mid z_i^{-1} z_i = 1$ ✓

Associativity: $z_i(z_j z_k) = (z_i z_j)(z_k)$ ✓

d) All $\mathbb{R}$ with $n_i \circledast n_j = \sqrt{n_i n_j}$

This is not a group as it violates closure. If either n_i or $n_j < 0$, then $\sqrt{n_i n_j} \notin \mathbb{R}$

e) All $z_i \in \mathbb{C}$ with $z_i \circledast z_j = \sqrt{z_i z_j}$

This also is NOT a group as it violates identity as the identity is not unique.

The pattern would be

$\sqrt{Ia} = a$

which only works if $I = a \rightarrow$ the identity is not unique!

HW #1 4A

4. Work out the symmetry group for a square: how many elements are there? Construct the multiplication table & determine if the group is Abelian.

Consider the square below

$$\boxed{\begin{matrix} a & b \\ d & c \end{matrix}}$$

There are 8 possible operations.

Identity $\boxed{\begin{matrix} a & b \\ d & c \end{matrix}}$

The three CCW rotations

$$\boxed{\begin{matrix} b & c \\ a & d \end{matrix}} \quad \boxed{\begin{matrix} c & d \\ b & a \end{matrix}} \quad \boxed{\begin{matrix} d & a \\ c & b \end{matrix}}$$

Or, I can flip horizontally, then rotate

$$\boxed{\begin{matrix} b & a \\ c & d \end{matrix}} \quad \boxed{\begin{matrix} a & d \\ b & c \end{matrix}} \quad \boxed{\begin{matrix} d & c \\ a & b \end{matrix}} \quad \boxed{\begin{matrix} c & b \\ d & a \end{matrix}}$$

Flipping vertically first does not yield a new state

$$\boxed{\begin{matrix} d & c \\ a & b \end{matrix}}$$

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HW #1 4B

Thinking of how I figured out the elements, I can write them in terms of a horizontal flip (aka about a vertical axis) I will call F and a 180° CCW rotation I'll call R. For example, a vertical flip is the same as $FRR = FR^2$. From here, I can build the multiplication table:

	I	R	R^2	R^3	F	FR	FR^2	FR^3
I	I	R	R^2	R^3	F	FR	FR^2	FR^3
R	R	R^2	R^3	I	FR	FR^2	FR^3	F
R^2	R^2	R^3	I	R	FR^2	FR^3	F	FR
R^3	R^3	I	R	R^2	FR^3	F	FR	FR^2
F	F	FR^3	FR^2	FR	I	R^3	R^2	R
FR	FR	F	FR^3	FR^2	R	I	R^3	R^2
FR^2	FR^2	FR	F	FR^3	R^2	R	I	R^3
FR^3	FR^3	FR^2	FR	F	R^3	R^2	R	I

Note, R^2, R^3, FR, FR^2, FR^3 are all unique group members as they are needed for closure as visible in the table above.

As the table is not symmetric, we can see that this group (D8) is non-Abelian

HW #1 5A

5 Show that the set of $n \times n$ unitary matricies constitutes a group.

Several of the properties are straightforward

✓ Associativity: Matrix multiplication is associative

✓ Identity: The $I_{n \times n}$ is in the set of unitary matricies. $I^+ I = I$

✓ Inverse: This is part of the definition of unitarity: $U^+ U = I_{n \times n}$

Thus, all we need to prove is closure.

Let U & V be two unitary matricies. Look at their product $P = UV$

Left multiply by $V^+ U^+$:

$$(V^+ U^+)(UV) = (V^+ U^+) P$$

$$V^+ U^+ U V = V^+ U^+ P$$

$$V^+ V = V^+ U^+ P$$

$$I = V^+ U^+ P$$

Since I, V^+, and U^+ are all in the group, P must be too!