

each page
in case they
come unstapled.

P564 SP'24

Number this way,
easy to know which
problem → IA

Homework 1

v.B.

For this first solution set, I will make some notes on best practices etc. that will help you get the full credit for writing quality as well as just general good habits. These are in red.

- Using $\langle x^2 \rangle$ and $\langle p^2 \rangle$, prove the virial theorem in the case of the QMHO.

$$\langle V \rangle = \langle T \rangle = \langle H \rangle / 2$$

It is good practice to write the problem, at least in summary. It makes it easier to follow & study later. I use blue for this to make it stand out.

We'll start with $\langle x^2 \rangle$:

$$\langle x^2 \rangle = \left\langle n \left| \frac{\hbar}{2m\omega} (a_-^2 + a_- a_+ + a_+ a_- + a_+^2) \right| n \right\rangle$$

The squared terms will always go to zero by orthogonality.

Note this explanation. Also note that I don't use more than 1 = sign per line. Equals means "equals" NOT "and then".

HW #1

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega} \langle n | (a_- a_+ + a_+ a_-) | n \rangle$$

Recalling that

$$a_+ |n\rangle = \sqrt{n+1} |n+1\rangle \text{ and } a_- |n\rangle = \sqrt{n} |n-1\rangle$$

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega} (\sqrt{n+1}\sqrt{n+1} + \sqrt{n}\sqrt{n})$$

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega} (n + (n+1))$$

The $\langle p^2 \rangle$ follows similarly:

$$\langle p^2 \rangle = \langle n | -\frac{m\hbar\omega}{2} (a_+^2 - a_- a_+ - a_+ a_- + a_-^2) | n \rangle$$

Again, all squared terms are zero as before

$$\langle p^2 \rangle = +\frac{m\hbar\omega}{2} \langle n | a_- a_+ + a_+ a_- | n \rangle$$

$$= \frac{m\hbar\omega}{2} (n + (n+1))$$

Now, for the virial theorem, we need $\langle V \rangle$

$$\langle V \rangle = \left\langle \frac{1}{2} m\omega^2 x^2 \right\rangle \xrightarrow{\text{Use an arrow for multiple steps}} \langle V \rangle = \frac{1}{2} m\omega^2 \langle x^2 \rangle \text{ on one line.}$$

HW #1

$$\langle V \rangle = \frac{1}{2} m \omega^2 \left[\frac{\hbar}{2m\omega} (2n+1) \right]$$

$$\langle V \rangle = \frac{\hbar \omega}{4} (2n+1)$$

Pulling out the 2 from the (2n+1)
 $\rightarrow 2(n + \frac{1}{2})$

$$\langle V \rangle = \frac{\hbar \omega \cdot 2}{4} (n + \frac{1}{2})$$

$$\langle V \rangle = \frac{\hbar \omega}{2} (n + \frac{1}{2}) = \frac{\langle H \rangle}{2}$$

Here, double equal are appropriate as my goal is to say the middle is equal to both.

Once we recall
 $\langle H \rangle = \hbar \omega (n + \frac{1}{2}) = \hbar \omega n$

We, of course, also need $\langle T \rangle$

$$\langle T \rangle = \left\langle \frac{p^2}{2m} \right\rangle$$

$$\langle T \rangle = \frac{1}{2m} \langle p^2 \rangle$$

$$\langle T \rangle = \frac{1}{2m} \left[\frac{m \hbar \omega}{2} (2n+1) \right]$$

HW #1

1D

$$\langle T \rangle = \frac{\hbar \omega}{4} (2n+1)$$

$$\langle T \rangle = \frac{1}{2} \hbar \omega (n + \frac{1}{2}) = \frac{\langle H \rangle}{2}$$

$$\therefore \langle V \rangle = \langle T \rangle = \frac{\langle H \rangle}{2} \quad \text{Q.E.D.}$$

HW #1

Starting each on a new page \rightarrow 2A

2. Show that, for $L(x_i, \dot{x}_i)$ ($\partial L / \partial t = 0$), a symmetry w.r.t. t ($t \rightarrow \delta t$) implies conservation of energy. You will need to assume $\partial L / \partial t = \partial L / \partial \dot{x}_i = 0$.

$$\frac{dL}{dt} = \frac{\partial L}{\partial x_i} \dot{x}_i + \frac{\partial L}{\partial \dot{x}_i} \ddot{x}_i + \frac{\partial L}{\partial t} \overset{0 \text{ by assumption}}{\rightarrow}$$

Recall the Euler-Lagrange equations

$$\frac{\partial L}{\partial x_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i}$$

which, after substitution, yields

$$\frac{dL}{dt} = \dot{x}_i \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} + \frac{\partial L}{\partial \dot{x}_i} \ddot{x}_i$$

N.B. I'm using the Einstein convention of repeated indices imply summation.

We can "un-chain-rule" to get

$$\frac{dL}{dt} = \frac{d}{dt} \left(\dot{x}_i \frac{\partial L}{\partial \dot{x}_i} \right)$$

$$\frac{dL}{dt} - \frac{d}{dt} \left(\dot{x}_i \frac{\partial L}{\partial \dot{x}_i} \right) = 0$$

HW #1

Pulling the time derivative outside:

$$\frac{d}{dt} \left(L - \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} \right) = 0$$

which implies

$$L - \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} = \text{constant}$$

Now, $L \equiv T - U$ so

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} - \frac{\partial U}{\partial \dot{x}_i} \rightarrow 0 \text{ by assumptions of the problem}$$

which leaves us with

$$L - \dot{x}_i \frac{\partial T}{\partial \dot{x}_i} = \text{Constant}$$

Eqn numbers are useful. Use them!
↓
(Eqn. 1)

Euler's theorem says that for any homogeneous function $f(y_k)$

$$y_k \frac{\partial f}{\partial y_k} = n f$$

where n is the degree of the function. In our case, the degree of T is $n=2$ ($T = \frac{1}{2} m \dot{x}_i^2$ after all!)

HW #1

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Thus, by Euler's Theorem

$$x_i \frac{\partial T}{\partial x_i} = 2T$$

Which, when substituted into Eqn. 1 along with $h \equiv T - U$ yields

$$(T - U) - 2T = \text{Constant}$$

$$-T - U = \text{Constant}$$

$$T + U = \text{Constant}$$

Where we recognize the constant as the total energy.

Q. E. D.

HW #1

3. Which of the following are groups

a) All real numbers under multiplication

This is NOT a group. The issue is inverse: the element zero does not have a real inverse which yields 1. I.e.

$$\nexists g_i \in \mathbb{R} \mid g_i \cdot 0 = \underline{1}$$

b) Real numbers under addition

Closure: $\forall \{g_i, g_k\} \in \mathbb{R}, g_i + g_k \in \mathbb{R} \checkmark$

Identity: Zero, which is in \mathbb{R} , is the identity: $\forall g_i \in \mathbb{R}, 0 + g_i = g_i$ (including when $g_i = 0$!) \checkmark

Inverse: $\forall g_i \in \mathbb{R} \exists g_i^{-1} = -g_i$ which is also in \mathbb{R} . Even zero works: $0^{-1} = -0, -0 + 0 = 0 \checkmark$

Associativity: $(a+b)+c = a+(b+c) \checkmark$

HW #1

- c) All complex (\mathbb{C}) numbers, except zero, under usual multiplication

Closure: If $\{z_i, z_j\} \neq 0$ and $\{z_i, z_j\} \in \mathbb{C}$, then $z_i z_j \in \mathbb{C}$ and $\neq 0$. ✓

Identity: $I = 1 \quad \forall z_i \in \mathbb{C}, 1 \cdot z_i = z_i$ ✓

Inverse: $\forall z_i \in \mathbb{C} \exists z_i^{-1} = 1/z_i \mid z_i^{-1} z_i = 1$ ✓

Associativity: $z_i (z_j z_k) = (z_i z_j) z_k$ ✓

- d) All \mathbb{R} with $n_i \otimes n_j = \sqrt{n_i n_j}$

This is not a group as it violates closure. If either n_i or $n_j < 0$, then $\sqrt{n_i n_j} \notin \mathbb{R}$

- e) All $z_i \in \mathbb{C}$ with $z_i \otimes z_j = \sqrt{z_i z_j}$

This also is NOT a group as it violates identity as the identity is not unique.

The pattern would be

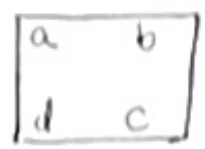
$$\sqrt{Ia} = a$$

which only works if $I = a \rightarrow$ the identity is not unique!

HW #1

4. Work out the symmetry group for a square: how many elements are there? Construct the multiplication table & determine if the group is Abelian.

Consider the square below



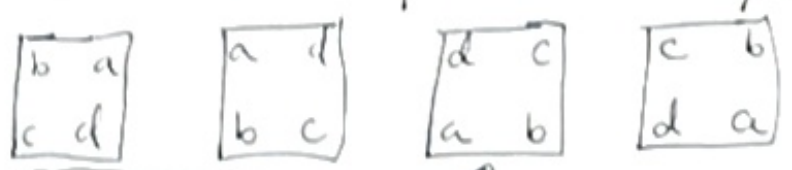
There are 8 possible operations.



The three CCW rotations



Or, I can flip horizontally, then rotate



Flipping vertically first does not yield a new state



HW#1

Thinking of how I figured out the elements, I can write them in terms of a horizontal flip (aka about a vertical axis) I will call F and a 180° CCW rotation I'll call R . For example, a vertical flip is the same as $FRR = FR^2$. From here, I can build the multiplication table:

	I	R	R^2	R^3	F	FR	FR^2	FR^3
I	I	R	R^2	R^3	F	FR	FR^2	FR^3
R	R	R^2	R^3	I	FR	FR^2	FR^3	F
R^2	R^2	R^3	I	R	FR^2	FR^3	F	FR
R^3	R^3	I	R	R^2	FR^3	F	FR	FR^2
F	F	FR^3	FR^2	FR	I	R^3	R^2	R
FR	FR	F	FR^3	FR^2	R	I	R^3	R^2
FR^2	FR^2	FR	F	FR^3	R^2	R	I	R^3
FR^3	FR^3	FR^2	FR	F	R^3	R^2	R	I

Note, R^2, R^3, FR, FR^2, FR^3 are all unique group members as they are needed for closure as visible in the table above.

As the table is not symmetric, we can see that this group (D_8) is non-Abelian

HW #1

5 Show that the set of $n \times n$ unitary matrices constitutes a group.

Several of the properties are straightforward

✓ Associativity: Matrix multiplication is associative

✓ Identity: The $I_{n \times n}$ is in the set of unitary matrices: $I^* I = I$

✓ Inverse: This is part of the definition of unitarity: $U^* U = I_{n \times n}$

Thus, all we need to prove is closure.

Let U & V be two unitary matrices. Look at their product $P = UV$

Left multiply by $V^* U^*$:

$$(V^* U^*) (UV) = (V^* U^*) P$$

$$V^* U^* U V = V^* U^* P$$

$$V^* V = V^* U^* P$$

$$I = V^* U^* P$$

Since I , V^* , and U^* are all in the group, P must be too!